

The Bogomolov multiplier of rigid finite groups

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Abstract. The Bogomolov multiplier of a finite group G is defined as the subgroup of the Schur multiplier consisting of the cohomology classes vanishing after restriction to all abelian subgroups of G . This invariant of G plays an important role in birational geometry of quotient spaces V/G . We show that in many cases the vanishing of the Bogomolov multiplier is guaranteed by the rigidity of G in the sense that it has no outer class-preserving automorphisms.

§1. Introduction

The main object of this note is the following invariant of a finite group G :

$$(1) \quad B_0(G) = \ker[H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \bigoplus_{A \subset G} H^2(A, \mathbb{Q}/\mathbb{Z})]$$

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where A runs over all abelian subgroups of G . Bogomolov showed in [Bo] that this group coincides with the unramified Brauer group $\text{Br}_{\text{nr}}(V/G)$ where V is a vector space defined over an algebraically closed field k of characteristic zero equipped with a faithful, linear, generically free action of G . The latter group is an important birational invariant of the quotient variety V/G , introduced by Saltman in [Sa1], [Sa2]. He used it for producing the first counter-example (for G of order p^9) to a problem by Emmy Noether on rationality of fields of invariants $k(x_1, \dots, x_n)^G$, where k is algebraically closed and G acts on the variables x_i by permutations. Formula (1) provides a purely group-theoretic intrinsic recipe for the computation of $\text{Br}_{\text{nr}}(V/G)$. In the same paper [Bo] Bogomolov showed that it can be simplified even further: one can replace A with the set of all *bicyclic* subgroups of G . In the case where G is a p -group, he also suggested a more explicit way for computing $B_0(G)$ and produced smaller counter-examples. For some further activity concerning the values of $B_0(G)$ for p -groups, as well as for corrigenda to some assertions of [Bo], the interested reader is referred to [CHKK], [HKK], [Mo1]. In particular, it turned out that the smallest power of p for which there exists a p -group G with $B_0(G) \neq 0$ is 5 (for odd p) and not 6, as claimed in [Bo]; see [Mo1] for details.

In the present paper, our viewpoint is a little different. Namely, we address the following question: what group-theoretic properties of G can guarantee that $B_0(G) = 0$? The first large family of groups (outside p -groups) for which we have $B_0(G) = 0$ is that of all *simple* groups [Ku1]. Thus, a natural question to ask is what common properties, shared by simple groups and “small” p -groups, are responsible for vanishing of $B_0(G)$. Our vague answer is that in a certain sense, both are *rigid*.

More precisely, the rigidity property we are talking about is the following one. Let a group G act on itself by conjugation, and let $H^1(G, G)$ be the first cohomology pointed set. Denote by $\text{III}(G)$ the subset of $H^1(G, G)$ consisting of the cohomology classes becoming trivial after restricting to every cyclic subgroup of G and call it the Shafarevich–Tate set of G (this terminology was introduced by T. Ono [On], alluding to arithmetic-geometric counterparts arising from the action of the Galois group of a number field k on the set of rational points of an algebraic k -group). We say that G is *III-rigid* if the set $\text{III}(G)$ consists of one element; see [Ku2] where this terminology was introduced in view of relationships with other rigidity properties of G . In the case where G is finite, $\text{III}(G)$ coincides with another local-global invariant $\text{Out}_c(G)$, which was introduced by Burnside [Bu1] about a century ago: it is the quotient of the group $\text{Aut}_c(G)$ of class-preserving automorphisms of G by the subgroup of inner automorphisms (an automorphism is called class-preserving if it moves each conjugacy class of G to itself). In particular, if G is finite, then $\text{III}(G)$ is a finite group, and G is *III-rigid* if and only if every locally inner automorphism $\varphi: G \rightarrow G$ (i.e., $\varphi(g) = aga^{-1}$ for some a depending on g) is inner (i.e., a can be chosen independent of g).

Certain classes of finite groups are known to consist of *III-rigid* groups. The following proposition collects some data from various sources.

Proposition 1.1 *The following finite groups are III-rigid:*

- (i) *symmetric groups* [OW];
- (ii) *simple groups* [FS];
- (iii) *p-groups of order at most p^4* [KV1];
- (iv) *p-groups having a cyclic maximal subgroup* [KV2];
- (v) *p-groups having a cyclic subgroup of index p^2* [KV3], [FN];
- (vi) *abelian-by-cyclic groups* [HJ];
- (vii) *groups such that the Sylow p -subgroups are cyclic for odd p , and either cyclic, or dihedral, or generalized quaternion for $p = 2$* [He1] (see [Su], [Wa] for a classification of such groups);
- (viii) *Blackburn groups* [He2], [HL];
- (ix) *extraspecial p -groups* [KV2];
- (x) *primitive supersolvable groups* [La];
- (xi) *unitriangular matrix groups over \mathbb{F}_p and the quotients of their lower central series* [BVY];
- (xii) *central products of III-rigid groups* [KV2].

See [Ya2] for a survey and some details.

Our main result states that the Bogomolov multiplier of most of the groups listed above is trivial.

Theorem 1.2 *Let G be one of the groups listed in items (i)–(ix) of Proposition 1.1. Then $B_0(G) = 0$.*

This theorem is proved in §2. Some open questions arising from this “experimental” observation are briefly discussed in §3.

Notational conventions. Unless otherwise stated, G denotes a finite group and k stands for an algebraically closed field of characteristic zero.

§2. Main results and proofs

We start the proof of Theorem 1.2 by observing that most of the work had already been done earlier. Namely, the assertions referring to items (i)–(vii) of Proposition 1.1 can be extracted from the literature, sometimes in a somewhat stronger form, stating that the relevant quotient varieties V/G are retract rational, stably rational, or even rational. Item (i) is a direct consequence of the classical theorem by Emmy Noether

asserting the rationality of the field of invariants $k(x_1, \dots, x_n)^{S_n}$ with respect to the natural permutation action of the symmetric group S_n (which follows from the theorem on elementary symmetric functions). The rationality of V/G is also known in cases (iii) [CK], (iv) [HK], (v) [Ka1]. In case (vi) the variety V/G is retract rational [Ka2], which is weaker than rationality but enough to guarantee vanishing of $B_0(G)$ [Sa2, Proposition 1.8]. The Bogomolov multiplier is zero in case (ii) [Ku1]. In case (vii), one can notice that in view of [Bo], [BMP], it is enough to establish that $B_0(S) = 0$ for all Sylow subgroups S of G . This is obvious for odd primes because in that case S is cyclic, and the groups appearing in the case $p = 2$ are all included in case (iv) above. Thus it remains to consider cases (viii) and (ix), which constitute the main body of the paper. They are treated separately below.

Proposition 2.1 *If G is an extraspecial p -group, then $B_0(G) = 0$.*

Before starting the proof, we present the following useful observation. Recall that groups G_1 and G_2 are called *isoclinic* if they have isomorphic quotients $G_i/Z(G_i)$ and derived subgroups $[G_i, G_i]$, and these isomorphisms are compatible (see, e.g., [Be, p. 285]).

Lemma 2.2 [Mo2] *If G_1 and G_2 are isoclinic, then $B_0(G_1) \cong B_0(G_2)$.*

Remark. The assertion of this lemma was stated in [HKK] as a conjecture. It was generalized in [BB] by showing that the quotient varieties V/G_i of isoclinic groups are stably birationally equivalent. Note also a striking parallel with a result of Yadav [Ya1], establishing the isomorphism $\text{III}(G_1) \cong \text{III}(G_2)$ for isoclinic groups.

Proof of Proposition 2.1. Recall that the centre Z of G is of order p and the quotient G/Z is a (nontrivial) elementary abelian p -group of order p^{2n} . There is a classification of such groups (see, e.g., [Go, pp. 203–208]). Since all elementary p -groups of the same order are isoclinic, in light of Lemma 2.2 we may and will consider only groups of exponent p . So from now on

$$(2) \quad G = \langle z, x_1, \dots, x_n, x_{n+1}, \dots, x_{2n} \mid [x_i, x_{i+n}] = z, i = 1, \dots, n \rangle$$

(all other generators commute and are all of exponent p).

Our computations are based on [Bo, Lemma 5.1] (we use the notation of [Pe, Section 5]). Namely, for a vector space E/\mathbb{F}_p we denote by E^\vee the dual space. We identify $\bigwedge^i(E^\vee)$ with $(\bigwedge^i E)^\vee$ and denote it by $\bigwedge^i E^\vee$. For any subset B in $\bigwedge^i E$ (resp. $\bigwedge^i E^\vee$) we denote by B^\perp its orthogonal in $\bigwedge^i E^\vee$ (resp. $\bigwedge^i E$). We view the abelian p -groups $Z = \langle z \rangle$ and $G/Z = \langle \bar{x}_i, i = 1, \dots, 2n \rangle$ as vector spaces over \mathbb{F}_p and denote them by V and U respectively (to ease the notation, we suppress bars over the x_i throughout below). Then we have the following central extension of vector spaces

$$0 \rightarrow V \rightarrow G \rightarrow U \rightarrow 0,$$

which gives rise to a surjective linear map $\gamma: \bigwedge^2 U \rightarrow V$ and the induced injective dual map $\gamma^\vee: V^\vee \rightarrow \bigwedge^2 U^\vee$. Let K^2 denote the image of γ^\vee , and let $S^2 = (K^2)^\perp \subset \bigwedge^2 U$.

Let S_{dec}^2 be the subgroup of S^2 generated by the decomposable elements of the form $u \wedge v$ ($u, v \in U$). Finally, let $K_{\text{max}}^2 \supset K^2$ be the orthogonal to S_{dec}^2 in $\bigwedge^2 U^\vee$. Then by [Bo, Lemma 5.1] we have an isomorphism $B_0(G) \cong K_{\text{max}}^2/K^2$.

In our case, we have

$$\gamma^\vee(\check{z}) = \sum_{i=1}^n \check{x}_i \wedge \check{x}_{i+n},$$

where $\check{\cdot}$ indicates to elements of the dual basis. Hence $S^2 \subset \bigwedge^2 U$ is the hyperplane

$$\left\{ \sum_{i < j} \alpha_{i,j} x_i \wedge x_j \mid \sum_{i=1}^n \alpha_{i,i+n} = 0 \right\}.$$

It is spanned by the elements $x_i \wedge x_j$ ($j > i$, $j \neq i+n$) and $x_i \wedge x_{i+n} - x_n \wedge x_{2n}$ ($i = 1, \dots, n-1$). Each of the latter elements can be represented in the form

$$x_i \wedge x_{i+n} - x_n \wedge x_{2n} = (x_i - x_n) \wedge (x_{i+n} + x_{2n}) - x_i \wedge x_{2n} + x_n \wedge x_{i+n},$$

i.e., as a sum of decomposable elements of S^2 . Hence each of the generators of S^2 belongs to S_{dec}^2 , and we have $S^2 = S_{\text{dec}}^2$, whence $K_{\text{max}}^2 = K^2$, so $B_0(G) = 0$. \blacksquare

This result can be extended to another class of groups, so-called almost extraspecial groups. Recall (see, e.g., [CT]) that a p -group G is called *almost extraspecial* if its centre $Z(G)$ is cyclic of order p^2 , and the Frattini subgroup $\Phi(G)$ coincides with the derived subgroup $[G, G]$ and they are both cyclic of order p . Any such group is of order p^{2n+2} , $n \geq 1$, and any two almost extraspecial groups of the same order are isomorphic.

Corollary 2.3 *If G is an almost extraspecial p -group, then $B_0(G) = 0$.*

Proof. The subgroup H of G generated by all elements of order p is extraspecial of order p^{2n+1} . If we denote by z a generator of $Z(G)$, then z^p can be taken as a generator of $Z(H)$, and we obtain compatible isomorphisms $G/Z(G) \cong H/Z(H)$ (both are elementary abelian of order p^{2n}) and $[G, G] \cong [H, H]$ (both are cyclic of order p), so G and H are isoclinic. The assertion of the corollary now follows from Proposition 2.1. \blacksquare

Proposition 2.4 *If G is a Blackburn group, then $B_0(G) = 0$.*

First recall the needed definitions.

Definition 2.5 A group G is called a *Dedekind group* if any subgroup of G is normal [Be, p. 33].

Remark. All Dedekind groups are classified [Be, pp. 33–34]. A Dedekind group is either abelian, or a direct product of a quaternion group of order 8 and an abelian group without elements of order 4. In both cases we have $B_0(G) = 0$.

Definition 2.6 A non-Dedekind group G is called a *Blackburn group* if the intersection of all its non-normal subgroups is nontrivial.

All such groups are classified [Bl], and in the proof below we proceed case by case.

Proof of Proposition 2.4. If G is a p -group, then, according to [Bl, Theorem 1], $p = 2$ and G is either a direct product of quaternion groups and abelian groups, or contains an abelian subgroup of index 2. In both cases, $B_0(G) = 0$, taking into account items (iii), (iv) above and the formula $B_0(G_1 \times G_2) = B_0(G_1) \times B_0(G_2)$ [Ka3].

So suppose that G is not a p -group. By [Bl, Theorem 2], there are five types of such groups. For types (a), (b), (d) and (e) the assertion is an immediate consequence of earlier considerations. Indeed, groups of types (a) and (d) are abelian-by-cyclic, and we use item (v) above. In case (b), G is a direct product of abelian and quaternion groups, and the argument of the preceding paragraph works. Groups of type (e) are direct products of quaternion, abelian, and abelian-by-cyclic groups, and we proceed as above.

It remains to consider case (c), where G contains a subgroup H of index 2 with the following property: H has an index two abelian subgroup A of exponent 2^k , k odd. Let S_p denote a Sylow p -subgroup of G . If p is odd, then S_p is abelian, hence $B_0(S_p) = 0$. Consider $S = S_2$. If S is a Dedekind group, then $B_0(S) = 0$ in light of the remark after Definition 2.5. If S is not a Dedekind group, then the intersection of its non-normal subgroups is nontrivial because each non-normal subgroup of S is a non-normal subgroup of G and G is a Blackburn group. So S is a Blackburn group too, and $B_0(S) = 0$ (see the first paragraph of the proof). Thus $B_0(S_p) = 0$ for all p , and therefore $B_0(G) = 0$ [Bo], [BMP] (see the first paragraph of the section). ■

Theorem 1.2 now follows from Propositions 2.1 and 2.4.

§3. Concluding remarks

We collect here several general remarks and open questions.

Question 3.1 Let G be a group belonging to class (x) or (xi) of Proposition 1.1. Is it true that $B_0(G) = 0$?

Here is a more general question:

Question 3.2 Let G be a III-rigid group. Is it true that $B_0(G) = 0$?

Note that there are groups G with $B_0(G) = 0$ that are not III-rigid. Say, so are first counter-examples to III-rigidity constructed by Burnside [Bu2]: these are groups of order 32 for which it is known that $B_0(G) = 0$ [CHKP].

Returning to the list of Proposition 1.1 and looking at the last item, we may ask the following parallel questions:

Question 3.3

- (i) Let $G = G_1 * G_2$ be a central product of groups such that $B_0(G_1) = B_0(G_2) = 0$. Is it true that $B_0(G) = 0$?
- (ii) Let $G = G_1 * G_2$ be a central product of groups such that the corresponding generically free linear quotients V_1/G_1 and V_2/G_2 are stably rational. Is it true that so is V/G ?

Definitely, it is much more tempting to understand whether there exists some intrinsic relationship between III-rigidity and Bogomolov multiplier behind the empirical observations presented in this paper. The interested reader is referred to [Ku2] for some speculations around these eventual ties.

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